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Cycle lengths and circuit matroids of graphs

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Abstract

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Let G and H be graphs without cut-edges. It is proved that there is a length-preserving group-isomorphism between the (integer) cycle groups of G and H if and only if G and H have isomorphic circuit matroids. It follows that a knowledge of the length function on the cycle group of G (up to isomorphism) conveys exactly the same amount of information about G as does a knowledge of its circuit matroid. This partially answers a question of O. Pretzel.

1. Introduction

Let G be a graph with edges $e_1, \ldots, e_m$; loops and multiple edges are permitted. For each edge e_i of G, choose one of the two orientations of e_i at random so as to form an oriented graph $\vec{G}$. The (first integer) *chain group* $C(\vec{G})$ of $\vec{G}$ is the free abelian group with the oriented edges $\vec{e}_1, \ldots, \vec{e}_m$ as generators. If $c_1, \ldots, c_m$ are integers, the element $c = \sum c_i\vec{e}_i$ of $C(\vec{G})$ is a *chain* on $\vec{G}$, which we write as $(c_1, \ldots, c_m)$, calling c_i the *coordinate* of $\vec{e}_i$ in c. The *length* $l(c)$ of c is defined by $l(c) := \sum_{i=1}^{m} |c_i|$.

For each circuit (connected 2-regular subgraph) C of G, choose one of the two orientations of C at random so that it defines a *circuit chain* in $C(\vec{G})$, in which each edge has coordinate 0, 1 or -1. Let $Z(\vec{G})$ be the subgroup of $C(\vec{G})$ generated by the circuit chains. In homology theory, $Z(\vec{G})$ is called the *first cycle group* of $\vec{G}$, and its elements are 1-*cycles, with integer coefficients*. We shall call $Z(\vec{G})$ simply the *cycle group* of $\vec{G}$, and its elements, *cycles*. If a chain is thought of as defining a flow on the edges of the graph, then it is a cycle if and only if the conservation law holds at each vertex; that is, the net flow out of each vertex is zero.

For example, if G is K_4 minus an edge, and α_1 and α_2 are appropriate circuit chains corresponding to the two triangles in G, then the cycle group of $\vec{G}$ is isomorphic to the free abelian group on two generators α_1 and α_2 with length function l satisfying

$$l(x\alpha_1 + y\alpha_2) = 2\,|x| + 2\,|y| + |x - y|.$$

Alternatively, writing $\beta_1 = \alpha_1 + \alpha_2$ and $\beta_2 = 2\alpha_1 + 3\alpha_2$ (which is not a circuit chain), the cycle group is isomorphic to the free abelian group on two generators β_1 and β_2 with

$$l(x\beta_1 + y\beta_2) = 2\,|x + 2y| + 2\,|x + 3y| + |y|.$$

Motivated by the main result of [3], Pretzel [2] asked to what extent the length function on the cycle group determines G. In particular, he conjectured that if G and H are 3-connected graphs and there is a length-preserving group-isomorphism between their cycle groups, then they are isomorphic graphs. The purpose of this paper is to prove the following.

Theorem 1. *Let G and H be graphs without cut-edges. Then there is a length-preserving group-isomorphism between the cycle groups of G and H if and only if G and H have isomorphic circuit matroids.*

It follows that the length function on the cycle group conveys exactly the same amount of information about a graph with no cut-edges as does its circuit matroid. This partially answers Pretzel's question, since the extent to which a graph is determined by its circuit matroid is fairly well understood. In particular, the truth of Pretzel's conjecture for 3-connected graphs follows from the following result, proved by Welsh [5, p. 83] as a slight generalization of a result of Whitney [6], and given with a shorter proof by Aigner [1, p. 352].

Theorem 2. *Let G and H be graphs with isomorphic circuit matroids such that G is 3-connected (without loops) and H has no isolated vertices. Then $G \cong H$.*

2. The proof of Theorem 1

We first prove that the circuit matroid of a graph determines the length function on its cycle group. This follows from the following theorem.

Theorem 3. *Let G and H be graphs with edges $e_1, \ldots, e_m$ and $f_1, \ldots, f_m$ respectively, such that for each subset I of $\{1, \ldots, m\}$, $\{e_i : i \in I\}$ is a circuit in G if and only if $\{f_i : i \in I\}$ is a circuit in H. Let $\{\vec{e}_1, \ldots, \vec{e}_m\}$ be an orientation of the edges of G. Then the edges of H can be oriented in such a way that, for each choice of $\varepsilon_1, \ldots, \varepsilon_m \in \{0, 1, -1\}$, $\sum \varepsilon_i \vec{e}_i$ is a circuit chain in $C(\vec{G})$ if and only if $\sum \varepsilon_i \vec{f}_i$ is a circuit chain in $C(\vec{H})$.*

Proof. We prove the result by induction on m, noting that it is obvious if $m=1$ or if every edge of G is a loop. So suppose that $m \geq 2$ and that some edge of G is not a loop, say $\vec{e}_m$ joins u to v where $v \neq u$. Form $G_1 := G/e_m$ and $H_1 := H/f_m$ by contracting e_m and f_m respectively, and let $\vec{G}_1$ be the orientation of G_1 induced by $\vec{G}$. If $C \subseteq \{e_1, \ldots, e_{m-1}\}$ then C is a circuit in G_1 if and only if C or $C \cup \{e_m\}$ is a circuit in G; and a similar remark holds for H_1. Thus G_1 and H_1 satisfy the hypothesis of the theorem, and so by the induction hypothesis there is an orientation $\vec{H}_1$ of H_1 with the required property. This induces an orientation of every edge of H except for f_m.

If there is no circuit in G containing e_m then we can orient f_m arbitrarily and the result will hold. Otherwise, let $z_1, \ldots, z_r$ be the circuit chains in $C(\vec{G})$ that have coordinate $+1$ on $\vec{e}_m$. Then, for each j, $z_j = y_j + \vec{e}_m$ where y_j is a circuit chain in $C(\vec{G}_1)$ that represents an oriented path from v to u in $\vec{G}$, and y_j corresponds to a circuit chain x_j in $C(\vec{H}_1)$ that represents an oriented path in $\vec{H}$ connecting one end of f_m to the other, so that either $x_j + \vec{f}_m$ or $x_j - \vec{f}_m$ is a circuit chain in $C(\vec{H})$. Choose the orientation $\vec{f}_m$ of f_m so that $x_1 + \vec{f}_m$ is a circuit chain, and suppose that, for some j, $x_j + \vec{f}_m$ is not a circuit chain. Then $x_j - \vec{f}_m$ is a circuit chain, and so $x_1 + x_j$ is a cycle in $Z(\vec{H}_1)$. Thus $y_1 + y_j$ must be a cycle in $Z(\vec{G}_1)$. But this is impossible, because $y_1 + y_j$ contains two edges entering u and no edge leaving u. So in fact $x_j + \vec{f}_m$ is a circuit chain in $C(\vec{H})$ for every j, whence the orientation $\vec{H}$ has the required property. □

To prove the converse, that the length function on the cycle group determines the circuit matroid of a graph, we first need some definitions and a lemma. Let G have edges $e_1, \ldots, e_m$ as usual. We can identify the m-tuple $(a_1, \ldots, a_m)$ with a set of edges of G if every a_i is 0 or 1, and with a multiset of edges if every a_i is a nonnegative integer. If $A = (a_1, \ldots, a_m)$ and $B = (b_1, \ldots, b_m)$ are multisets, we write $A \subseteq B$ if $a_i \leq b_i$ for each i. If $c = (c_1, \ldots, c_m)$ is a chain in $C(\vec{G})$, let $M(c)$ denote the multiset of (unoriented) edges of G defined by $M(c) := (|c_1|, \ldots, |c_m|)$. Let $\mathscr{C} := \{M(c) \colon c \in Z(\vec{G})\}$. The following lemma is implicit in Theorem 1.22 of Tutte [4]; I include the proof here for completeness.

Lemma 4.1. *The minimal (under $\subseteq$) non-empty multisets in $\mathscr{C}$ are all sets, which are precisely the same as the circuits of G.*

Proof. Every circuit C belongs to $\mathscr{C}$, since $C = M(c)$ where c is the circuit chain corresponding to an orientation of C. Thus to prove the lemma it suffers to prove that if $z \in Z(\vec{G})$ and $M(z) \neq \emptyset$, then there is a circuit C such that $C \subseteq M(z)$. Let G_z be the subgraph of G consisting of all edges on which $M(z)$ has nonzero coordinate, together with their incident vertices. Then G_z contains no vertex of degree 1 (this is most easily seen by thinking of z as a flow), and so G_z contains a circuit C. Then $C \subseteq M(z)$, as required. □

We are now in a position to prove that the length function on the cycle group determines the circuit matroid of a graph, from which Theorem 1 immediately follows.

Theorem 4. *Suppose we are given an abelian group Γ, to every element α of which there is assigned a nonnegative integer $l(\alpha)$ called its* length. *Suppose that there is a length-preserving group-isomorphism between Γ and the cycle group of some graph G that has no cut-edges (but we do not know G or the isomorphism, merely that they exist). Then we can construct, up to matroid isomorphism, the circuit matroid of G.*

Proof. We first show how to identify the elements of Γ that correspond to circuit chains of $\vec{G}$ under the isomorphism, and hence to multisets of edges that are circuits of G. Let α and β be nonzero elements of Γ that correspond to cycles a and b with corresponding multisets $A = M(a)$ and $B = M(b)$. Then by considering a and b it is easy to see that $B \subseteq A$ if and only if $l(\alpha+\beta) + l(\alpha-\beta) = 2l(\alpha)$. Thus the length function on Γ determines which elements α of Γ correspond to multisets A that are minimal non-empty members of the collection $\mathscr{C}$, and hence circuits by Lemma 4.1.

In fact, if $B \subseteq A$ and $B \neq A$ then $l(\beta) < l(\alpha)$, and for fixed α there are only finitely many elements β of Γ satisfying this inequality, so that it is a finite process to check whether α corresponds to a circuit. Also, one can easily find an upper bound for the set of lengths $l(\alpha)$ such that α corresponds to a circuit—for example, one (poor) upper bound is the sum of the lengths of a set of generators of Γ—and so there are only finitely many elements of Γ that need to be checked in order to determine all the circuits.

Now let $\alpha_1, \ldots, \alpha_s$ be all the elements of Γ that correspond to circuits, let $z_1, \ldots, z_s$ be the corresponding circuit chains and $C_1, \ldots, C_s$ the corresponding circuits. (There is actually no point in including both α and $-\alpha$ in the list, but it does no harm to do so.) For each j, we know the cardinality of C_j: $|C_j| = l(\alpha_j)$. Now, it is easy to see that

$$l(\alpha_1) + l((k-2)\alpha_1 - \alpha_2 - \cdots - \alpha_k) - l((k-1)\alpha_1 - \alpha_2 - \cdots - \alpha_k)$$

is twice the number of edges of $\vec{G}$ on which $z_1, \ldots, z_k$ all have the same nonzero coordinate (1 or -1). It follows that

$$2\,|C_1 \cap C_2 \cap \cdots \cap C_k| = 2^{k-1} l(\alpha_1) + \sum \big(l((k-2)\alpha_1 + \varepsilon_2\alpha_2 + \cdots + \varepsilon_k\alpha_k) - l((k-1)\alpha_1 + \varepsilon_2\alpha_2 + \cdots + \varepsilon_k\alpha_k)\big)$$

where each ε_j is ± 1 and the sum ranges over all 2^{k-1} possible choices of sign. Thus we can calculate the cardinality of the intersection of any collection of the circuits C_j. Since G has no cut-edges, every edge of G lies in at least one of the circuits. We can now construct the circuit matroid of G, since a Venn diagram of the circuits has 2^s cells and we can calculate how many edges lie in each of these cells, thus determining the circuit matroid (up to matroid isomorphism). This completes the proof of Theorem 4, and with it the proof of Theorem 1. □

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